Perfectly Matched Anisotropic Layer for the Numerical Analysis of Unbounded Eddy-Current Problems

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***Abstract*—An extension of the perfectly matched layer (PML) technique in quasi-static fields is developed. The new low-frequency PML is based on a fictitious medium with diagonal tensor anisotropy. On the basis of a theoretical investigation, the material properties of the anisotropic layer are specified, so that it will be reflectionless for an arbitrary eddy-current field that may exist in free space. Furthermore, the PML is designed in such a way that outgoing eddy-current fields are sufficiently absorbed. The effectiveness of the low-frequency PML is validated by the implementation of the finite-element solution of a simple two-dimensional eddy-current problem as well as a more complicated three-dimensional one.**

***Index Terms*—Eddy currents, edge elements, numerical methods, perfectly matched layers.**

I. Introduction

THE perfectly matched layer (PML) technique, which was introduced by Berenger [1] for the free-space simulation of unbounded two-dimensional (2-D) high frequency problems, is based on the concept of the use of a fictitious layer designed to absorb outgoing waves without causing any reflection. Katz *et al.* [2] validated this technique and extended it to three-dimensional (3-D) finite difference time domain (FDTD) grids. However, this kind of PML, which is reflectionless for all frequencies, polarizations, and angles, involves a modification of Maxwell's equations. Hence, its implementation requires a modification of the standard FDTD equations, using the concept of split field components.

Sacks *et al.* [3] proved that the reflectionless properties of the PML may also be obtained by an appropriate anisotropic medium. In contrast to the conventional PML approach, this one, also known as Maxwellian PML, does not require any modification of Maxwell's equations. Furthermore, it is easily implemented in any existing finite element or FDTD code that deals with anisotropic materials and provides a better physical understanding of the PML [4].

So far, the PML technique having been introduced as a promising alternative to the concept of absorbing boundary conditions [5]–[6], has been almost exclusively implemented in high frequency applications, such as scattering or radiation problems [7]–[8].

Manuscript received December 8, 1998; revised July 15, 1999.

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Publisher Item Identifier S 0018-9464(99)09363-2.

In static or quasi-static field problems, the truncation of the infinite space is usually performed by means of a hybrid finite element-boundary element formulation ([9]–[13]), in which the unbounded free space is modeled by an integral equation. Although exact, this approach results in partially dense matrices and the demands in storage and computation may be high. Recently, an appropriate modification of the Maxwellian PML technique has been proposed for the solution of open boundary static field problems, so that the size of the mesh, and thus the computational burden, will be minimized, as Bardi *et al.* [14] and Ticar *et al.* [15] have shown.

It is the purpose of this paper to introduce a new anisotropic absorbing layer, appropriate for mesh truncation of open-boundary eddy-current problems. The methodology of constructing a PML for quasi-static fields is conceptually different from that in the case of high frequency problems, where it is required that the reflection coefficient for a wave incident on the interface between air and the absorbing layer is zero. In the case of low frequency fields, the derivation of PML properties will be based on the fulfillment of matching conditions for the fields on the interface, whereas it is *a priori* enforced that the presence of the layer will not affect their values in the domain of interest. The whole procedure could be considered as an extension of the approach for electrostatic fields [14]. The resulting PML can be used to efficiently terminate finite element meshes in the case of eddy-current problems. Unlike hybrid methods, all the characteristics of a differential formulation, like sparsity of the system matrices and the use of iterative methods for system solution, are preserved in our approach. The new scheme is validated by the implementation of a finite element solution of a simple 2-D skin and proximity effect problem ([16]–[18]) and that of a 3-D one, known as Nakata's conductor [19].

II. Formulation and Properties of the PML in Low Frequencies

A. Electric Field Formulation

For the investigation of the properties of a PML in the quasi-static field case, we adopt first a simplified 2-D model, which will be later extended in the 3-D case. We also assume an electric field formulation.

Let us suppose that the domain of interest Ω_{int} is truncated by a rectangular boundary Γ (Fig. 1). The free space simu-

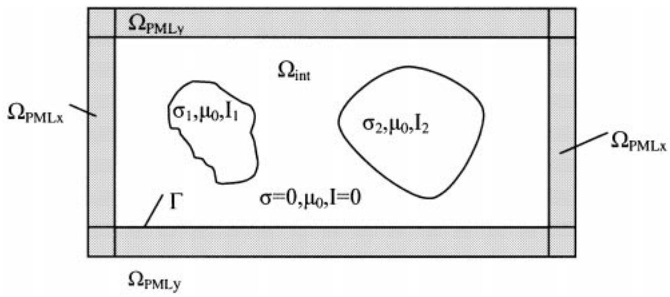


Fig. 1. General topology of the 2-D quasi-static case.

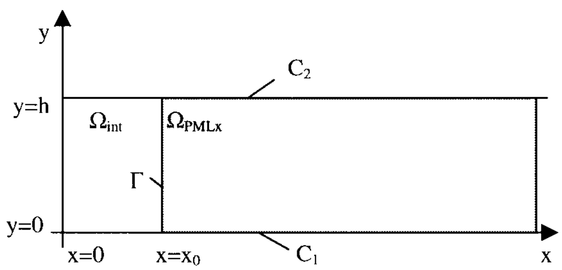


Fig. 2. The PML for truncation along the $\boldsymbol{x}$ direction (PMLx).

lation is obtained through the introduction of an absorbing boundary with appropriate properties. To simplify the problem we consider only one side of the rectangular region (Fig. 2).

Two cases are considered. In the first case the strip is filled with vacuum, while in the second one the PML is introduced, with the interface separating it from the domain of interest being located at $x = x_0$. Thus, the PML occupies the area $x \geq x_0, 0<y<h$.

The governing equations of the problem, when the displacement term is neglected, are

$$\nabla \times \boldsymbol{E} = -\overline{\mu}\frac{\partial \boldsymbol{H}}{\partial t} = -j\omega\overline{\mu}\boldsymbol{H} \tag{1}$$

$$\nabla \times \boldsymbol{H} = \overline{\sigma}\boldsymbol{E} \tag{2}$$

where we assumed that the domain of interest is source-free ($\boldsymbol{J_s} = 0$), that there is harmonic time variation, and that the magnetic permeability $\overline{\mu}$ and the electric conductivity $\overline{\sigma}$ are diagonal, time-invariant tensors, that is

$$\overline{\mu} = \mu_0 \begin{bmatrix} \mu_x & 0 & 0 \\ 0 & \mu_y & 0 \\ 0 & 0 & \mu_z \end{bmatrix} \quad \text{and} \quad \overline{\sigma} = \begin{bmatrix} \sigma_x & 0 & 0 \\ 0 & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{bmatrix}.$$

It is noted that in vacuum $\mu_x = \mu_y = \mu_z = 1$ and $\sigma_x = \sigma_y = \sigma_z = 0$, whereas the PML is an anisotropic medium, and hence the values of the tensor elements are not necessarily equal to each other. Combining (1) and (2), we get the following equation, which stands in vacuum as well as in the PML region:

$$\nabla \times \overline{\mu}^{-1}\nabla \times \boldsymbol{E} = -j\omega\overline{\sigma}\boldsymbol{E}. \tag{3}$$

For the 2-D case we assume that only the z component of the electric field intensity $\boldsymbol{E}$ is nonzero, that is $\boldsymbol{E} = E(x,y)\hat{\boldsymbol{z}}$. Therefore, (3) can be written in the form

$$\frac{1}{\mu_y}\frac{\partial^2 \boldsymbol{E}}{\partial x^2} + \frac{1}{\mu_x}\frac{\partial^2 \boldsymbol{E}}{\partial y^2} = j\omega\sigma_z\mu_0 E. \tag{4}$$

First of all we will investigate the general form of solutions for (4) both in free space and the PML. Using a standard analytical approach, a general solution to (4) is given by

$$E = e^{-k_x\sqrt{\mu_y}x}\left(c_1 e^{-jk_y\sqrt{\mu_x}y} + c_2 e^{jk_y\sqrt{\mu_x}y}\right) \tag{5}$$

whereas the magnetic field intensity H is calculated via (1) and is given by

$$\begin{aligned} \boldsymbol{H} = \frac{j}{\omega}\Bigg\{ & j\frac{k_y}{\sqrt{\mu_x}}e^{-k_x\sqrt{\mu_y}x}\left(-c_1 e^{-jk_y\sqrt{\mu_x}y} + c_2 e^{jk_y\sqrt{\mu_x}y}\right)\hat{\boldsymbol{x}} \\ & + \frac{k_x}{\sqrt{\mu_y}}e^{-k_x\sqrt{\mu_y}x}\left(c_1 e^{-jk_y\sqrt{\mu_x}y} + c_2 e^{jk_y\sqrt{\mu_x}y}\right)\hat{\boldsymbol{y}}\Bigg\} \end{aligned} \tag{6}$$

where k_x and k_y satisfy the relation

$$k_x^2 - k_y^2 = j\omega\sigma_z\mu_0. \tag{7}$$

For the PML analysis we assume that an arbitrary eddy-current field exists within free space. We require that its values remain unchanged when the PML is placed (Fig. 2). This requirement is similar to the property of zero reflection in the high frequency case. Unlike previous approaches in the static field case we do not consider the existence of boundaries C_1: $y = 0, x \geq x_0$ and C_2: $y = h, x \geq x_0$. For example, the electrostatic potential is supposed to be zero on C_1, C_2 in [14], which is a rather restrictive condition. A general case is presented in [15] where the field is expanded on C_1, C_2 by means of a Fourier series. In our approach we consider a PML in the semi-infinite space $x \geq x_0$, without the presence of C_1, C_2, since they will affect only the absorption rate within the layer. The electric field in free space, according to (5), will be given by

$$E^{\text{air}} = e^{-kx}\left(c_1 e^{-jky} + c_2 e^{jky}\right)$$

since $k_x = k_x = k$ due to (7) and the fact that $\sigma = 0$. Similarly, the magnetic field intensity is given by

$$\begin{aligned} \boldsymbol{H}^{\text{air}} = \frac{i}{\omega}\big\{ & jke^{-kx}\left(-c_1 e^{-jky} + c_2 e^{jky}\right)\hat{\boldsymbol{x}} \\ & + ke^{-kx}\left(c_1 e^{-jky} + c_2 e^{jky}\right)\hat{\boldsymbol{y}}\big\} \end{aligned}$$

According to (5) and (6), the fields inside the PML are given by

$$E^{\text{PML}} = e^{-k_x\sqrt{\mu_y}x}\left(c_1' e^{-jk_y\sqrt{\mu_x}y} + c_2' e^{jk_y\sqrt{\mu_x}y}\right)$$

and

$$\begin{aligned} \boldsymbol{H}^{\text{PML}} = \frac{j}{\omega}\Bigg\{ & j\frac{k_y}{\sqrt{\mu_x}}e^{-k_x\sqrt{\mu_y}x} \\ & \cdot\left(-c_1' e^{-jk_y\sqrt{\mu_x}y} + c_2' e^{jk_y\sqrt{\mu_x}y}\right)\hat{\boldsymbol{x}} \\ & + \frac{k_x}{\sqrt{\mu_y}}e^{-k_x\sqrt{\mu_y}x} \\ & \cdot\left(c_1' e^{-jk_y\sqrt{\mu_x}y} + c_2' e^{jk_y\sqrt{\mu_x}y}\right)\hat{\boldsymbol{y}}\Bigg\}. \end{aligned}$$

The matching conditions for the fields on the interface are the continuity of the tangential component of the electric and magnetic field intensity, respectively,

$$E^{\text{air}}|_{x=x_0} = E^{\text{PML}}|_{x=x_0}$$
$$H_y^{\text{air}}|_{x=x_0} = H_y^{\text{PML}}|_{x=x_0}$$

and the continuity of the normal component of the magnetic flux density

$$\mu_0 H_x^{\text{air}}|_{x=x_0} = \left(\overline{\mu} H^{\text{PML}}\right)_x|_{x=x_0}$$

which yield the following constraints:

$$k_y = \frac{k}{\sqrt{\mu_x}} \tag{8.a}$$
$$k_x = k\sqrt{\mu_y} \tag{8.b}$$
$$c_1' = c_1 \frac{e^{-kx_0}}{e^{-k_x\sqrt{\mu_y}x_0}} \tag{8.c}$$
$$c_2' = c_2 \frac{e^{-kx_0}}{e^{-k_x\sqrt{\mu_y}x_0}}. \tag{8.d}$$

If constraints (8.a) and (8.b) are combined with (7) we obtain

$$k^2\left(\mu_y - \frac{1}{\mu_x}\right) = j\mu_0\sigma_z\omega. \tag{8.e}$$

This equation should hold for every possible k, i.e., for any arbitrary field that can exist in free space. Hence

$$\mu_x\mu_y = 1 \tag{9.a}$$

and

$$\sigma_z = 0. \tag{9.b}$$

Thus, in order to have an efficient attenuation of the electric field along the $\boldsymbol{x}$ direction (normal to the interface between vacuum and the PML), the properties of the anistropic absorbing medium must be defined by (9.a) and (9.b).

Furthermore, the attenuation factor of the field within the layer is given by

$$R = e^{-k_x\sqrt{\mu_y}x} = e^{-k\mu_y x}. \tag{10}$$

Our aim is to obtain as much attenuation as possible, so the exponent in (10) must be large enough. Thus, the value of μ_y must exceed unity and, therefore, in accordance to (9.a), that of μ_x must be less than unity. Furthermore, the absorbing layer must be thick enough to cause the desirable attenuation of the field.

The extension of this procedure to the 3-D case is not straightforward, since a general analytical treatment, taking into account the three field components E_x, E_y, and E_z would result in coupled set of three ordinary differential equations involving nine unknown functions, which is very difficult to solve. Instead of this, we consider a restricted treatment taking into account one component at a time. This method covers very wide classes of fields, although some fields, which cannot be absorbed, may exist, as it will be shown.

If the z component of the electric field intensity $E_z(x,y,z)$ is considered, it is easily shown that the condition

$$\nabla \cdot \boldsymbol{E} = 0$$

gives

$$\frac{\partial E_z}{\partial z} = 0$$

which leads to the case $E_z(x,y)$ that has already been treated. In a similar way, if we consider the case $E_y(z,x)$ we obtain the conditions

$$\mu_x\mu_z = 1$$

and

$$\sigma_y = 0.$$

However, if we attempt to deal with the case $E_x(y,z)$, which has no dependence on the x coordinate, it can be shown that such fields cannot be absorbed by the PML if $\mu_y \neq 1, \mu_z \neq 1$. This case could be considered similar to a wave with grazing incidence, i.e., with a 90° angle of incidence, in the high frequency case. Therefore, although in three dimensions there are certain classes of fields not absorbed by the PML, its properties that ensure the attenuation of the field along the $\boldsymbol{x}$ direction should be as follows:

$$\overline{\mu} = \mu_0 \begin{bmatrix} \mu_x & 0 & 0 \\ 0 & 1/\mu_x & 0 \\ 0 & 0 & 1/\mu_x \end{bmatrix} \tag{11.a}$$

and

$$\overline{\sigma} = \begin{bmatrix} \sigma_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{11.b}$$

The value of σ_x can be arbitrary. Since it has no effect on the problem, we choose $\sigma_x = 0$, setting therefore the conductivity of the absorbing medium to be zero.

The PML proposed for the finite element analysis of eddy-current problems by an $\boldsymbol{E}$ formulation is also appropriate for the treatment of eddy-current problems by a magnetic vector potential formulation ($\boldsymbol{A}$ formulation). This is concluded by the similarity that the two formulations have. Specifically we get $\boldsymbol{A}$ formulation, if we replace $\boldsymbol{E}$ with $\boldsymbol{A}$ in (3).

B. Magnetic Field Formulation

It could be expected that because of the duality of the electromagnetic field quantities, the properties of a PML appropriate for the treatment of eddy-current problems by an $\boldsymbol{H}$ formulation would arise from the transposition of $\overline{\mu}$ and $\overline{\sigma}$ in the $\boldsymbol{E}$ formulation PML. However, due to the null value of air conductivity and since we have neglected the displacement current, this duality does not hold in the quasi-static field. Therefore, the matching conditions on the interface between air and the absorbing layer cannot be satisfied using the concept of duality.

Alternatively, using the concept of the magnetic scalar potential, we can easily prove that the PML for a magnetic field formulation in the quasi-static field is the same as the one for the magnetostatic field [15], i.e., the magnetic permeability

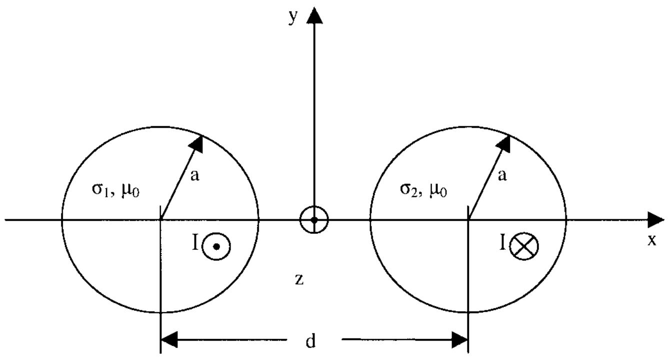


Fig. 3. Geometry of the 2-D multiconductor system.

tensor is given by

$$\overline{\mu} = \mu_0 \begin{bmatrix} \mu_x & 0 & 0 \\ 0 & 1/\mu_x & 0 \\ 0 & 0 & 1/\mu_x \end{bmatrix}$$

while the conductivity tensor is zero.

III. Numerical Results

In order to validate the scheme that was proposed in this paper, some numerical results were carried out. Two eddy-current problems, in 2-D and in 3-D, respectively, were elaborated by a Galerkin $\boldsymbol{E}$ formulation. In the 2-D problem the efficiency of the PML with an $\boldsymbol{E}$ formulation is tested in comparison with a reference solution, whereas in the 3-D problem the new scheme is used for the computation of the eddy currents in a multiply connected conductor.

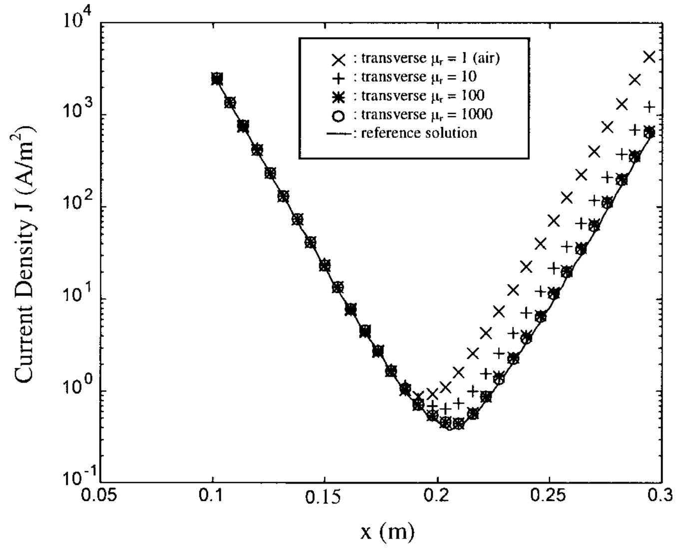
A. Eddy Currents in a Multiconductor System

The geometry of the problem is depicted in Fig. 3. The system consists of two circular current carrying conductors of conductivity σ_1 and σ_2 and radius a. The total current of each conductor is sinusoidal at 50 Hz frequency and of constant amplitude I. It can be proved that the current density in each conductor is composed of three terms, the first, $\boldsymbol{J_0}$, denoting the uniform distribution of current I, and the two others, $\boldsymbol{J_1}$ and $\boldsymbol{J_2}$, denoting the eddy currents induced by the magnetic field of the first conductor and the magnetic field of the second conductor, respectively. Therefore, the total current density inside each conductor is expressed as the vectorial sum of the three terms

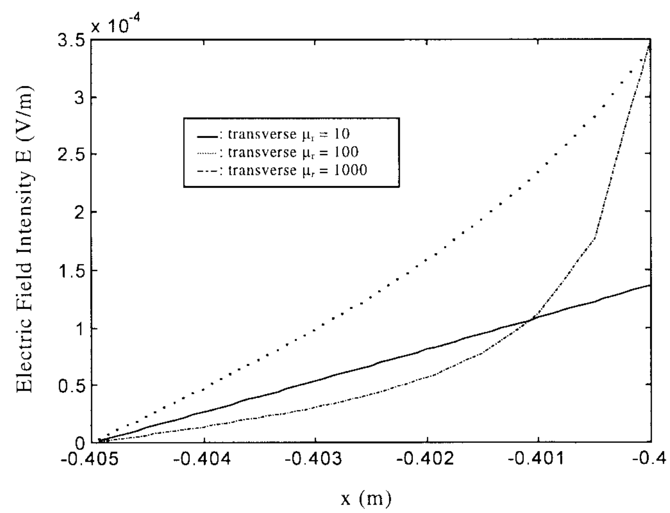
$$\boldsymbol{J} = \boldsymbol{J_0} + \boldsymbol{J_1} + \boldsymbol{J_2}.$$

The total current density was numerically computed by the use of conventional triangular nodal finite elements. A reference solution, the accuracy of which is ensured by the simplicity of the problem and the reliability of the finite element method, was obtained by setting the outer boundary far enough from the two conductors. The nodes laid on this outer boundary were enforced to a null value of the electric field intensity (Dirichlet boundary condition).

The solution of the problem with the PML as an attenuation medium has shown that, similar to the static field case [15], the two factors that control the PML performance are the thickness of the layer and the value of the conductivity and permeability tensor elements. The former must be such that the electric field reduces evenly; however it must be composed of an adequate number of elements, so as no errors emerge. It came up by the experiments that a large value of the transverse relative permeability is needed for a small thickness PML, whereas a small value of the transverse relative permeability is needed for a large thickness PML.



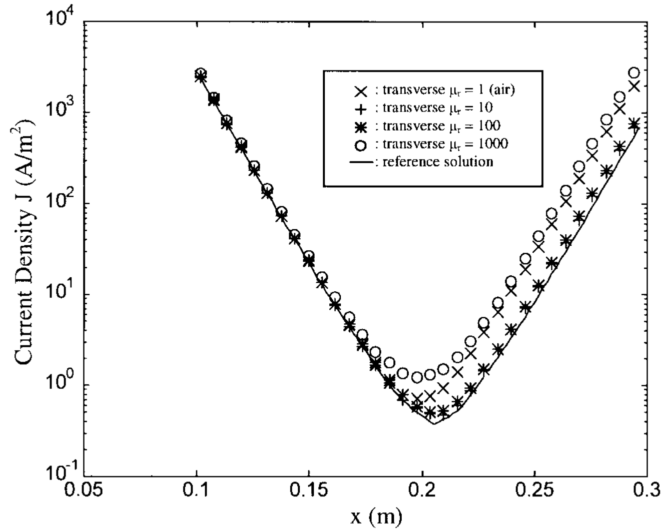
(a)



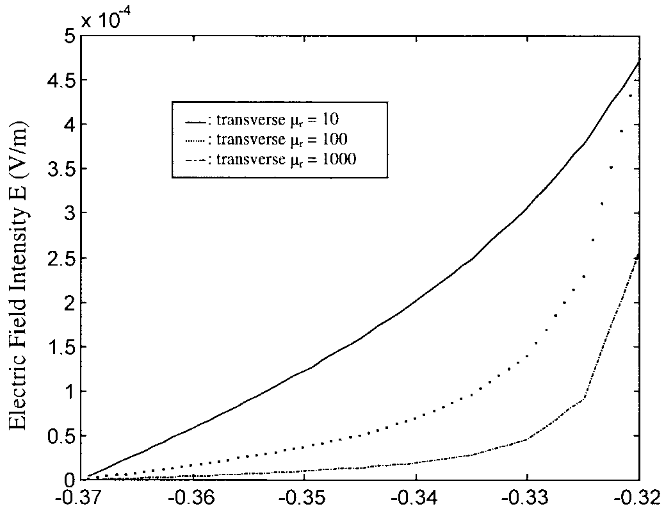
(b)

Fig. 4. (a) Current density distribution inside the right conductor (5 mm thick PML). (b) Attenuation of the electric field inside the PML region (5 mm thick PML).

In Fig. 4(a) the current density distribution inside the right conductor along the line $y = 0$ is depicted for different values of the transverse relative permeability of a 5 mm thick PML. Fig. 4(b) compares the attenuation of the magnitude of the electric field intensity inside the PML for the three cases. Similar results are depicted in Fig. 5 for a 50 mm thick PML. Obviously the results obtained for the 5 mm thick PML with a larger value of the transverse relative permeability are closer to the reference solution as opposed to the case of the 50 mm thick PML. The decay of the field is satisfactory in both cases,

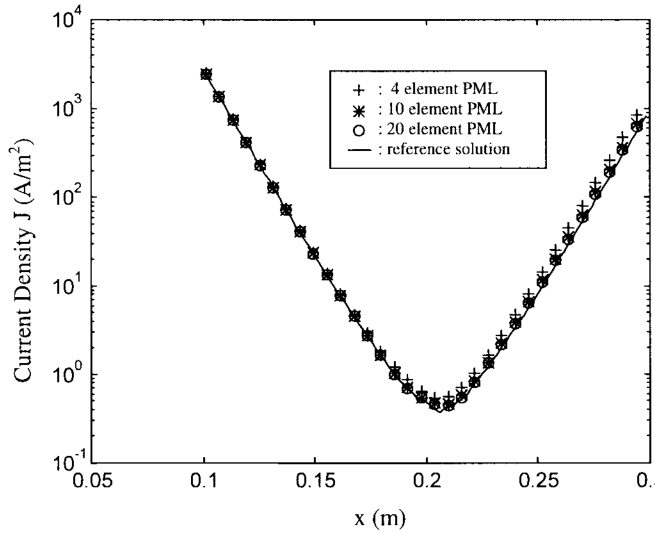


(a)

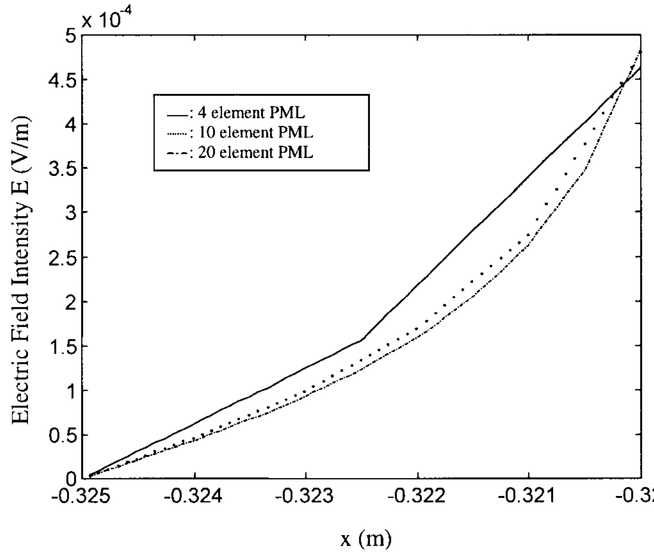


(b)

Fig. 5. (a) Current density distribution inside the right conductor (50 mm thick PML). (b) Attenuation of the electric field inside the PML region (50 mm thick PML).



(a)



(b)

Fig. 6. (a) Current density distribution inside the right conductor (5 mm thick PML — transverse relative permeability equal to 200). (b) Attenuation of the electric field inside the PML region (5 mm thick PML — transverse relative permeability equal to 200).

however in the latter, significant errors are introduced because of the larger element size.

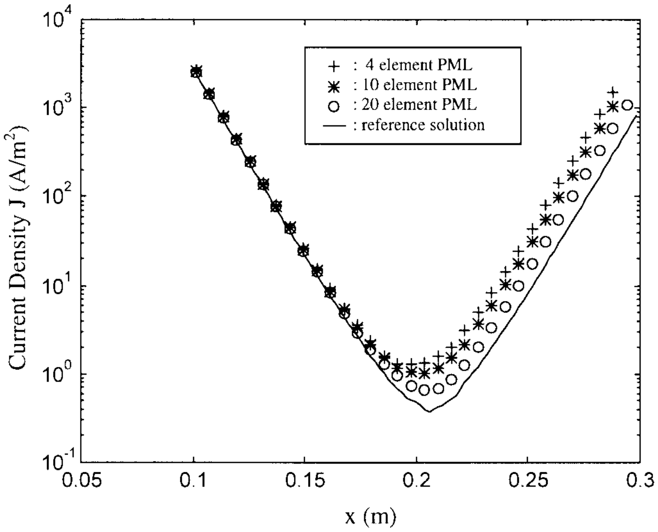
In Fig. 6(a) the current density distribution inside the right conductor is also depicted. Three cases are considered accordingly to the number of elements composing the PML. The value of the transverse relative permeability is 200 (that of the normal relative permeability is 0.005) and the PML is 5 mm thick. As it can be easily seen, both the accuracy of the solution and the attenuation of the electric field intensity inside the PML [Fig. 6(b)], increase with the number of the elements inside the PML region. Similar results are depicted in Fig. 7 for a 50 mm thick PML.

B. Eddy Currents Inside a Conductor with a Hole

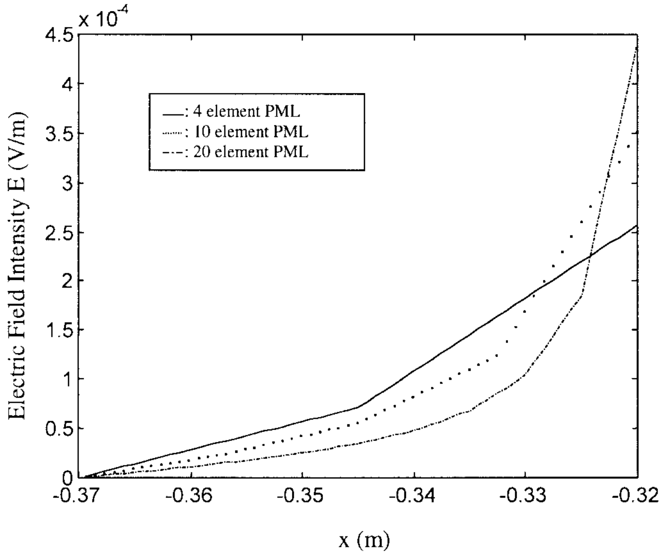
The problem's structure is depicted from two different points of view in Fig. 8(a) and (b). Although it is a quite complicated problem and has no analytical solution, Nakata's conductor has been a test problem for the evaluation of various finite element formulations in the past [19].

The conductor is multiply connected and therefore the use of electric or magnetic scalar potential ($\boldsymbol{E}-\varphi$ or $\boldsymbol{H}-\varphi$ formulation) requires special treatment [12], [13]. On the contrary $\boldsymbol{E}$ formulation is much more preferable, since it can be applied unconstrained. Therefore, a formulation based on the Galerkin weighted residual procedure in terms of the electric field intensity $\boldsymbol{E}$ and the corresponding PML, were chosen, along with edge elements. It is noted that at corners where two or three different layers intersect a combination of the properties of each layer is preferred [8].

Figs. 9 and 10 verify the deductions of the 2-D case as far as the value of the transverse relative permeability concerns. It is observed that a PML with transverse $\mu_r = 10$ approaches

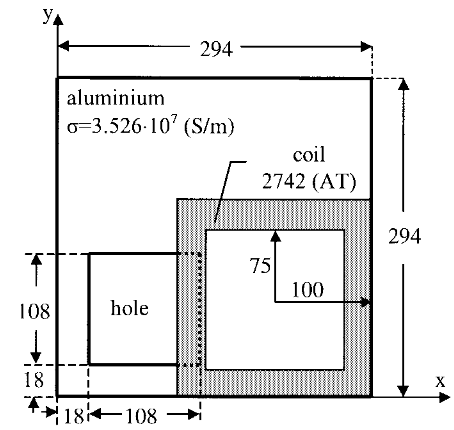


(a)

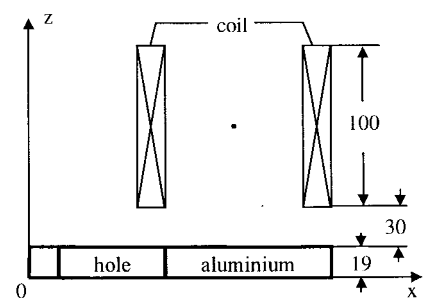


(b)

Fig. 7. (a) Current density distribution inside the right conductor (50 mm thick PML – transverse relative permeability equal to 200). (b) Attenuation of the electric field inside the PML region (50 mm thick PML – transverse relative permeability equal to 200).



(a)



(b)

Fig. 8. (a) Top view of the problem structure (dimensions in mm). (b) Transverse cut of the problem structure (dimensions in mm).

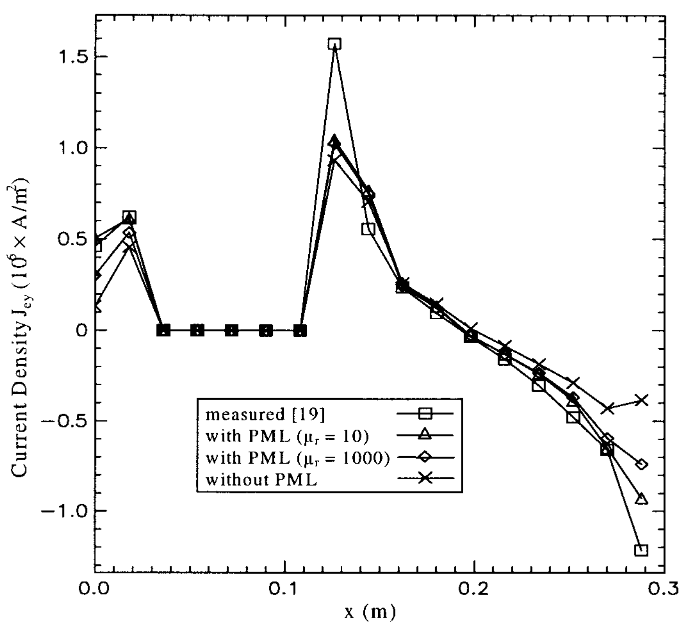


Fig. 9. y component of the eddy-current density along line $y = 72$ mm, $z = 19$ mm with the use of a 50–mm thick PML.

the experimental measurements better than one with transverse $\mu_r = 1000$. It is also obvious that these two cases are more effective than the non-PML case. Finally, Fig. 11 investigates the effectiveness of the proposed scheme as a function of the PML thickness.

IV. Conclusions

This paper introduces a new PML suitable for the numerical computation of open boundary quasi-static field problems. This PML is obtained by properly selecting the values of the permeability and conductivity tensor elements. It has been shown, by the implementation of the new scheme by a finite element solution of two simple skin and proximity effect problems, that the thickness of the PML, the values of the permeability and conductivity tensor elements, and the number of elements inside the PML region, are the main factors that control its efficiency. However, they are not independent. The accuracy obtained is much better if the above elements are combined in such a way that the electromagnetic field attenuates evenly introducing no significant errors. Finally,

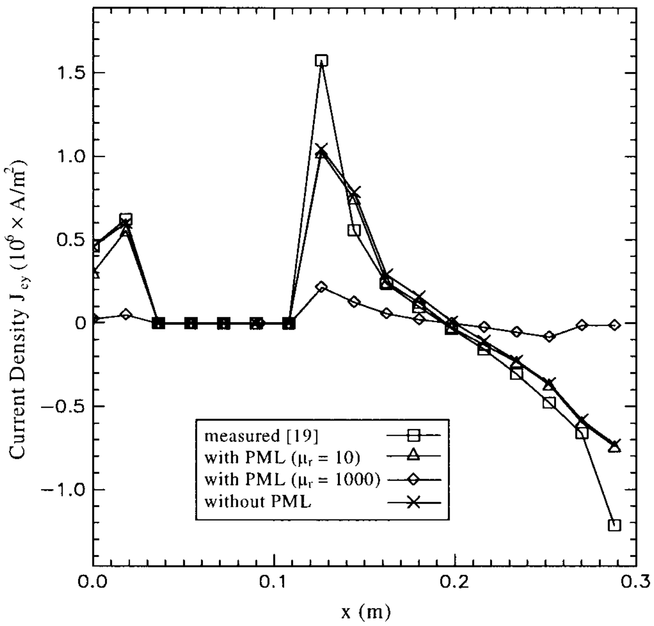


Fig. 10. y component of the eddy-current density along line $y = 72$ mm, $z = 19$ mm with the use of a 10–mm thick PML.

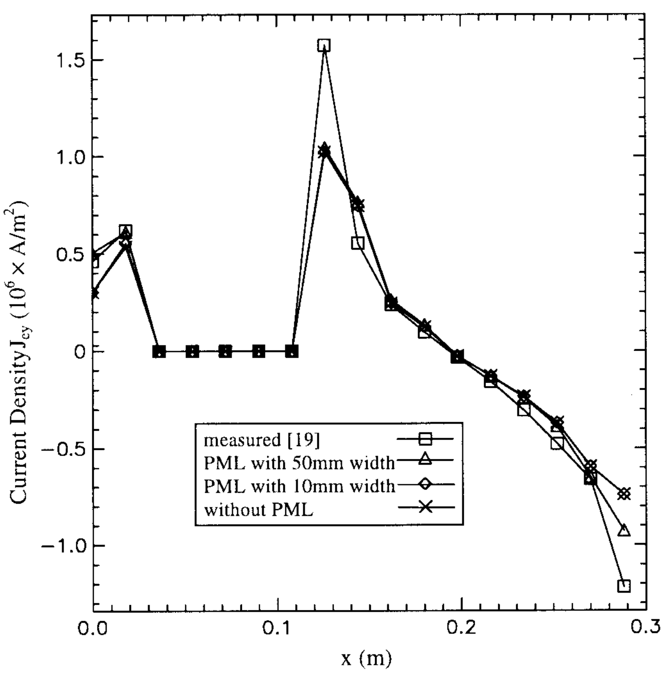


Fig. 11. y component of the eddy-current density along line $y = 72$ mm, $z = 19$ mm (transverve relative permeability equal to ten).

the new PML can be feasibly and effectively applied to the computation of eddy currents in complicated 3-D problems as it has been shown. The reduction of the computational burden obtained in the latter case is a significant gain. Hence, the PML could be a promising tool in the finite element analysis of eddy-current problems, where hybrid methods seemed to be the only reliable technique for mesh truncation.

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